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EXACT SHELL THEORY ANALYSIS OF FREE VIBRATIONS OF SUBMERGED THIN SPHERICAL SHELLS

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Abstract—By the introduction of two displacement functions, the non-axisymmetric free vibrations of a complete thin isotropic spherical shell submerged in a compressible fluid medium are successfully investigated. It is found that there exist two classes of free vibrations: the first class is not affected by the ambient fluid while the second is. It is further proved that the frequency equations can be expressed in terms of polynomial. For the second class, it is also demonstrated that only complex frequencies exist except for the case of $n = 1$, for which the trivial solution $\Omega = 0$ emerges. For $n = 0$, 1 and 2, the frequency equations of the second class are investigated numerically and the effects of various relative parameters are discussed. The small damping coefficient method is also discussed finally in the paper. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The vibration problems of spherical shells coupled with ambient media have attracted much attention both from engineers and scientists because of the wide usage of spherical shells in practical engineering. Junger (1952) first considered the free vibrations and the associated radiation of sound of a thin spherical shell submerged in a compressible fluid medium. Hayek (1966) studied the axisymmetric non-torsional vibration of a spherical shell in an acoustic medium by utilizing the Lagrange equation; as an example, he gave the analysis of dynamic behavior of a thin spherical shell submerged in a compressible fluid medium which undertook a point harmonic force. Lou and Su (1978) pointed out that the damp characteristic of free vibrations of submerged thin spherical shells was solely due to the compressibility of the fluid when the effect of the viscosity of the fluid was not taken into consideration, but strict demonstration was not given; for a steel spherical shell submerged in water, they calculated the frequencies with small damping components. Felippa and Geers (1980) carried the Laplace transform on the wave equation to obtain the fluid pressure and the frequency equation for axisymmetric case was derived with its root-loci plots presented. Su (1982) further discussed the effect of the fluid viscosity on the free vibrations of a submerged spherical shell, using a boundary layer approximation for the fluid medium. To the authors' knowledge, the available studies on the coupling vibrations of submerged spherical shells all dealt with the axisymmetric cases. However, the non-axisymmetric free vibrations of spherical shells in vacuum have been exhaustively studied by several authors (Prasad, 1964; Wilkinson and Kalnins, 1965; Ramakrishnan and Shah, 1970). Prasad (1964), for example, simplified the governing equations of a thin spherical shell to two uncoupled partial differential equations and one partly coupled partial differential equation by introducing some certain auxiliary variables and solutions in terms of associated Legendre functions were obtained. Recently, Ding and Chen (1996a, b) exactly studied the free vibrations of embedded or submerged spherically isotropic spherical shells based on the three-dimensional elastic theory; for the submerged case (Ding and Chen, 1996b), due to the inclusion of the second kind of spherical Hankel functions in the resulted frequency equations, the small damping coefficient method suggested by Lou and Su (1978) was employed to calculated the frequencies with small damping components. However,

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detailed and delicate investigation on the frequency equation that corresponds to the thin shell theory was not presented.

Lou and Su (1978) was the first to mention that the frequency equation of a submerged spherical shell can be written in a rational expression and the free vibration of the spherical shell in compressible fluid is damping. Felippa and Geers (1980) employed a different method to derive the rational form frequency equation, however, in their further discussion, they only considered the numerator polynomial and omitted the nominator one without strict demonstration. They showed the damp characteristic by root-loci plots also without theoretical verification.

This paper presents a detailed investigation on the thin shell theory which employs the Kirchhoff assumptions. First, by using relative formulae in Ding and Chen (1996b), the governing equations of nonaxisymmetric free vibrations of a submerged spherical shell are given. It is proven that there are two classes, however, only the second class is affected by the ambient fluid. Second, the rational form frequency equations are obtained and it is strictly demonstrated that the frequency equations only have complex roots, thus the damp characteristic of the second class free vibration is proven. It is further verified that the numerator and nominator polynomials have no common zero point, then the polynomial form frequency equations are consequently derived. The effects of various parameters are discussed. Finally, the practicability of small damping coefficient method is evaluated. Because the polynomial form frequency equation can be exactly solved, it can be used to clarify the precision of various approximate theories.

2. BASIC EQUATIONS AND DERIVATIONS

The spherical coordinates and the geometry of a thin spherical shell are shown in Fig. 1. Assume the middle surface displacement to be u, v and w in θ, ϕ and r direction, respectively. Allowing for the Kirchhoff assumptions, one has, in eqn (44) in Ding and Chen (1996b), $\beta_z = 0$ as well as

$$\beta_\theta = \frac{1}{R}\left(u - \frac{\partial w}{\partial \theta}\right), \quad \beta_\phi = \frac{1}{R}\left(v - \csc\theta \frac{\partial w}{\partial \phi}\right). \tag{1}$$

The governing equations can also be obtained by utilizing Hamilton's principle. According to eqn (46) in Ding and Chen (1996b), only the decomposition formulae of u and v are retained in this paper. Setting $RG_1 = \Phi$, one obtains

$$(\nabla_1^2 + \Omega^2 + 2)\Psi = 0, \tag{2}$$

$$(\nabla_1^2 + c_1)\Phi + (c_2\nabla_1^2 + c_3)w = 2\Psi R\cos\theta + \frac{1+\nu}{2}\frac{\partial\Psi}{\partial\theta}R\sin\theta, \tag{3}$$

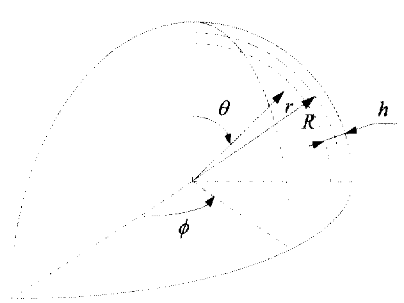

Fig. 1. Spherical coordinates and the geometry of a thin spherical shell.

$$\nabla_1^2\Phi+(c_4\nabla_1^4+c_5\nabla_1^2+c_6)w = 2\Psi R\cos\theta+\frac{\partial\Psi}{\partial\theta}R\sin\theta, \tag{4}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial\theta^2}+\cos\theta\frac{\partial}{\partial\theta}+\csc^2\theta\frac{\partial^2}{\partial\phi^2}, \tag{5}$$

$$\begin{aligned}
&c_1 = l_1(\Omega^2+2), \quad c_2 = -e^2/12\\
&c_3 = 1+v+2l_1c_2, \quad c_4 = -c_2/(1+v)\\
&c_5 = -2l_2c_2, \quad c_6 = 2-l_2\Omega^2+l_2l_3\Omega h_n(\Omega/c_0)\\
&\Omega = R\omega/v_2, \quad v_2^2 = E/[2(1+v)\rho]\\
&\rho_0 = \rho_f/\rho, \quad c_0 = c_f/v_2\\
&l_1 = (1-v)/2, \quad l_2 = l_1/(1+v)\\
&l_3 = \rho_0c_0/e, \quad e = h/R
\end{aligned} \tag{6}$$

and function $h_n(x)$ is defined by eqn (23) in Ding and Chen (1996b).

Obviously, eqn (2) is a second-order homogeneous partial differential equation only about function Ψ. It is shown that the radial displacement is absent. It actually corresponds to the first class of vibration in the three dimensional elastic theory (Ding and Chen, 1996b). Equations (3) and (4) are coupled by functions Φ and w, and it can be seen that the right hands of these two equations still include function Ψ. In fact, by virtue of eqn (2), functions Ψ and Φ can be eliminated skilfully from eqns (3) and (4) and a sixth-order partial differential equation only of function w can be obtained as follows

$$[c_4\nabla_1^6+(c_5+c_1c_4-c_2)\nabla_1^4+(c_6+c_1c_5-c_3)\nabla_1^2+c_1c_6]w = 0. \tag{7}$$

Though eqn (7) is only relative to function w, it is seen from eqns (3) and (4) that function Φ still exists. It is noticed here that eqn (7) corresponds to the second class of vibration in the three dimensional elastic theory (Ding and Chen, 1996b).

3. FREQUENCY EQUATIONS AND INVESTIGATION

For a complete spherical shell, it can be seen from eqns (2) and (7) that functions w and Ψ can be assumed in the following form

$$w = \sum_{n=0}^{\infty} A_nS_n^m(\theta,\phi)\exp(i\omega t), \quad \Psi = \sum_{n=0}^{\infty} B_nS_n^m(\theta,\phi)\exp(i\omega t), \tag{8}$$

where $S_n^m(\theta,\phi) = P_n^m(\cos\theta)\exp(im\phi)$ is the spherical harmonic function and $P_n^m(x)$ is the associated Legendre function of variable x; A_n and B_n are arbitrary constants. Substituting eqn (8) into eqns (2) and (7), respectively, one obtains two classes of frequency equations as

$$\Omega^2 = (n-1)(n+2), \tag{9}$$

$$c_4n^3(n+1)^3-(c_5+c_1c_4-c_2)n^2(n+1)^2+(c_6+c_1c_5-c_3)n(n+1)-c_1c_6 = 0. \tag{10}$$

It is seen that eqn (9) does not include any fluid parameter and it is identical to that obtained by Love (1944). We will not discuss eqn (9) later anymore.

For $n = 0$, the spherical shell has radial displacement only, and this mode is always called the breathing mode. In this case, eqn (9) has no meaning and only the second class exists, of which the frequency equation can be easily obtained as follows

$$c_6 = 0. \tag{11}$$

We rewrite eqns (11) and (10) in the following forms

$$F_1 c_0^2 x^2 - 2 = F_2 c_0 x h_0(x), \quad (n = 0), \tag{12}$$

$$F_3 c_0^4 x^4 + F_4 c_0^2 x^2 + F_5 = (F_6 c_0^2 x^2 + F_7) c_0 x h_n(x), \quad (n = 1, 2, \ldots), \tag{13}$$

where $x = \Omega/c_0$; F_i are nondimensional quantities defined as follows

$$\begin{aligned}
&F_1 = l_2, \quad F_2 = l_2 l_3, \quad F_3 = l_1 l_2 \\
&F_4 = 2l_1 l_2 - l_2 n(n+1) - l_1[c_4 n^2(n+1)^2 - c_5 n(n+1) + 2] \\
&F_5 = c_4 n^3(n+1)^3 + (c_2 - c_5 - 2l_1 c_4) n^2 (n+1)^2 + (2 - c_3 - 2l_1 c_5) n(n+1) - 4l_1 \\
&\quad = c_4(n-1)(n+2)[n(n+1)(n^2+n+\nu-1) + 2l_1/c_4] \\
&F_6 = l_1 F_2, \quad F_7 = [2l_1 - n(n+1)]F_2.
\end{aligned} \tag{14}$$

It is noted here that, if setting $h_n(x) = 0$, eqns (12) and (13) then become to those in the vacuum. Equation (12) will read in this case

$$\Omega = 2\sqrt{\frac{1+\nu}{1-\nu}}. \tag{15}$$

The above equation is identical to that obtained by Love (1944). If the spherical is very thin, i.e. $e \to 0$, then setting $c_2 = 0$ as well as $h_n(x) = 0$ in (13) gives

$$\Omega^4 - \frac{2}{1-\nu}(n^2+n+1+3\nu)\Omega^2 + 4(n^2+n-2)\frac{1+\nu}{1-\nu} = 0. \tag{16}$$

This equation is also identical to that obtained by Love (1944). If the ambient fluid is considered as incompressible, then the term $c_0 h_n(x)$ in eqns (12) and (13) should be replaced by $-\Omega/(n+1)$ (Ding and Chen, 1996b).

By virtue of the famous Rayleigh formula of spherical Hankel function (Watson, 1966)

$$h_n^{(2)}(x) = (-1)^n \mathrm{i} x^n \left(\frac{\mathrm{d}}{x\,\mathrm{d}x}\right)^n \left[\frac{1}{x}\exp(-\mathrm{i}x)\right], \quad (n = 0, 1, 2, \ldots), \tag{17}$$

one can rewrite $h_n^{(2)}(x)$ in the following form:

$$h_n^{(2)}(x) = \frac{A_n(x)}{x^{n+1}}\exp(-\mathrm{i}x), \quad (n = 0, 1, 2, \ldots), \tag{18}$$

where $A_n(x)$ represents an n-th order polynomial of variable x. By employing the recurrence formula of spherical Hankel function $h_n^{(2)}(x)$, one can derive the recurrence formula of polynomial $A_n(x)$ as

$$A_n(x) = (2n-1)A_{n-1}(x) - x^2 A_{n-2}(x), \quad (n = 2, 3, \ldots). \tag{19}$$

In addition, one can directly obtain

$$A_0(x) = i, \quad A_1(x) = \mathrm{i} - x. \tag{20}$$

From eqn (19), one gets

$$A_2(x) = 3A_1(x) - x^2 A_0(x) = 3(\mathrm{i} - x) - \mathrm{i}x^2. \tag{21}$$

By virtue of eqns (19) and (20), one can also ge the constant term of polynomial $A_n(x)$

$$a_0^0 = \mathrm{i}, \quad a_n^0 = 1 \cdot 3 \cdot 5 \ldots (2n-1)\mathrm{i}, \quad (n = 1, 2, \ldots). \tag{22}$$

Using eqn (18), the function $h_n(x)$, which were given by eqn (23) in Ding and Chen (1996b), can be rewritten in the following form

$$h_n(x) = xA_n(x)/B_{n+1}(x), \quad (n = 0, 1, 2, \ldots), \tag{23}$$

where

$$B_{n+1}(x) = nA_n(x) - A_{n+1}(x), \quad (n = 0, 1, 2, \ldots). \tag{24}$$

Obviously $B_{n+1}(x)$ is an $(n+1)$-th order polynomial of variable x with its constant term

$$b_{n-1}^0 = na_n^0 - (2n+1)a_n^0 = -(n+1)a_n^0, \quad (n = 0, 1, 2, \ldots). \tag{25}$$

Note that eqn (23) directly gives $h_n(0) = 0$, and so $x = 0$ (i.e. $\Omega = 0$) is not the root of eqn (12). If $x = 0$ is just the root of eqn (13), it demands $F_5 = 0$. It is seen from eqn (14) that this will be satisfied for $n = 1$. Because $l_1 > 0$ and $c_4 > 0$, one has $F_5 > 0$ when $n \geqslant 2$, $x = 0$ is no longer the root of eqn (13) for all $n > 1$.

It can be further shown that both eqns (12) and (13) have no non-zero real roots. First, by virtue of the Wronskian (Watson, 1966)

$$j_n(x)n_n'(x) - j_n'(x)n_n(x) = x^{-2}, \tag{26}$$

where $j_n'(x)$ and $n_n'(x)$ are the derivatives of the first and second kinds of spherical Bessel functions $j_n(x)$ and $n_n(x)$, respectively, one can get the imaginary part of function $h_n(x)$ as follows

$$\mathrm{Im}\,[h_n(x)] = \frac{1}{x^2(j'^2_n + n'^2_n)} > 0, \quad \text{(when } x \text{ is a non-zero real).} \tag{27}$$

Therefore, if eqns (12) and (13) have any real root, one should get, respectively,

$$F_2 = 0, \quad F_1 c_0^2 x^2 - 2 = 0, \tag{28}$$

$$F_6 c_0^2 x^2 + F_7 = 0, \quad F_3 c_0^4 x^4 + F_4 c_0^2 x^2 + F_5 = 0. \tag{29}$$

However, from eqn (14), one has $F_2 = l_2 l_3 > 0$. That is to say, eqn (28) cannot be satisfied any longer and, consequently, eqn (12) has no real root. For any real root $x \neq 0$ to exist for eqn (13), the following equation must be satisfied from eqn (29) by eliminating variable x

$$F = F_3F_7^2+F_4F_7F_6+F_5F_6^2 = 0. \tag{30}$$

But further calculation gives

$$F = -[1+c_4(n^2+n+v-1)](1+v)n(n+1)l_1^2F_2^2. \tag{31}$$

Since $c_4 > 0, 0 \leqslant v \leqslant 0.5$ and $n \geqslant 1$, one has $F < 0$. Obviously, this result is in) conflict with eqn (30). It means that eqn (13) has not yet non-zero real root.

By virtue of eqn (23), the frequency eqns (12) and (13) can be written in the rational form

$$C_3(x)/B_1(x) = 0, \quad (n = 0), \tag{32}$$

$$C_{n+5}(x)/B_{n+1}(x) = 0, \quad (n = 1, 2, \ldots), \tag{33}$$

where $C_n(x)$ represents a n-th order polynomial of variable x, one has

$$C_3(x) = (F_1c_0^2x^2-2)B_1(x)-F_2c_0x^2A_0(x), \quad (n = 0), \tag{34}$$

$$C_{n+5}(x) = (F_3c_0^4x^4+F_4c_0^2x^2+F_5)B_{n+1}(x)-(F_6c_0^2x^2+F_7)c_0x^2A_n(x), \quad (n = 1, 2, \ldots). \tag{35}$$

From eqns (20) and (24), on has $B_1(x) = x-\mathrm{i}$ and $C_3(\mathrm{i}) = \mathrm{i}c_0F_2 \neq 0$. It is obvious that they have no common zero point (or common factor), and thus, eqn (32) can be simplified to

$$C_3(x) = F_1c_0^2x^3-\mathrm{i}(F_1c_0^2+F_2c_0)x^2-2x+2\mathrm{i} = 0, \quad (n = 0). \tag{36}$$

Equations (20) and (21) directly show that $A_2(x)$ and $A_1(x)$ have no common zero point. One can further verity that $A_n(x)$ and $A_{n-1}(x)$ have no common zero point yet by employing the inductive method. The process can be simply stated here: if $A_n(x)$ and $A_{n-1}(x)$ have a common zero point $x_1 \neq 0$ [since $a_n^0 \neq 0$, see eqn (22)], then from eqn (19), x_1 is also the zero point of $A_{n-2}(x)$, however, this is conflict with the assumption of the inductive method. In a similar manner, one can verify that $B_{n+1}(x)$ and $A_n(x)$ have also no common zero point by virtue of eqn (24).

According to the results obtained above, one can only discuss the complex roots of frequency eqn (33) here and after. From eqn (35), the condition that $C_{n+5}(x)$ and $B_{n+1}(x)$ have a common complex zero point demands $F_6c_0^2x^2+F_7 = 0$. However, the solution this equation reads

$$x^2 = \frac{-F_7}{c_0^2F_6} = \frac{n(n+1)+v-1}{c_0^2l_1} > 0, \tag{37}$$

i.e. the root is a real. This means $C_{n-5}(x)$ and $B_{n+1}(x)$ cannot have any common complex zero point. So, in the case of seeking complex roots, the frequency eqn (33) can be simplified to

$$C_{n+5}(x) = 0, \quad (n = 1, 2, \ldots). \tag{38}$$

It can be further seen from eqn (35) that $x = 0$ is the double root of eqn (13) or (33) when $n = 1$, that is, all roots of original frequency eqns (13) and (33) can be solved from eqn (38). Thus, eqn (38) is quite equivalent to frequency eqn (13) or (33). As shown earlier above, all roots of eqn (12) or (32) can also be solved from eqn (36).

Therefore, through strict derivation, we transform the rational form frequency eqns (32) and (33) to the polynomial form frequency eqns (36) and (38) that include complex coefficients. In fact, one can further transform these resulted frequency equations to those only including real coefficients, the process is now described in the following.

Writing $A_n(x)$ in the form

$$A_n(x) = p_n(x) + \mathrm{i}q_n(x), \tag{39}$$

where $p_n(x)$ and $q_n(x)$ are both real polynomials. Their recurrence formulae are identical to that of $A_n(x)$. Further, from eqn (20), one gets

$$p_0 = 0, \quad q_0 = 1; \quad p_1 = -x, \quad q_1 = 1. \tag{40}$$

By virtue of their recurrence formulae, it is shown that $p_n(x)$ is an odd function, while $q_n(x)$ is an even one. This indicates that the coefficients of the odd order terms of polynomial $A_n(x)$ are real while those of the even ones are pure imaginary. From eqns (24), (34) and (35), it is known that $C_3(x)$ and $C_{n+5}(x)$ also have the same characteristic as that of $A_n(x)$. Therefore, setting

$$x = y\mathrm{i}, \tag{41}$$

and substituting it into eqns (36) and (38), one can get one cubic real algebraic equation and one real algebraic equation of $(n+5)$ degree, respectively,

$$R_3(y) = F_1 c_0^2 y^3 - (F_1 c_0^2 + F_2 c_0) y^2 + 2y - 2 = 0, \quad (n = 0), \tag{42}$$

$$R_{n+5}(y) = 0, \quad (n = 1, 2, \ldots). \tag{43}$$

As in the case studied by Silbiger (1962), the integer m, which appears in the spherical harmonic and represents the non-axisymmetric motion ($m \neq 0$) of the shell is not included in the frequency equations.

4. NUMERICAL RESULTS

It is shown that the nondimensional frequency Ω are related to nondimensional parameters ρ_0, c_0, v and e. In the following, we will investigate the effects of these parameters on the frequencies. Equation (36) and (42) is a cubic algebraic equation, its roots can be directly written out; to solve the roots of eqn (38) or (43), the Laguerre method is employed (Stoer and Bulirsch, 1980).

For different mode number n, the natural frequencies of the following four cases are calculated: (1) $e = 0.03$, $c_0 = 0.3$, $v = 0.3$ and with $\rho_0 = 0.1$–0.4; (2) $e = 0.03$, $\rho_0 = 0.2$, $v = 0.3$ and with $c_0 = 0.2$–0.5; (3) $e = 0.03$, $\rho_0 = 0.2$, $c_0 = 0.3$ and with $v = 0.1$–0.4; and (4) $v = 0.3$, $\rho_0 = 0.2$, $c_0 = 0.3$ and with $e = 0.01$–0.04. According to the numerical results, it is easy to draw the corresponding root-loci plots (to save space, figures are omitted in this paper). Results show that the root-loci plots are axisymmetric about the imaginary axis and the imaginary parts of complex frequencies are always greater than or equal to zero. This is actually identical to the practical situation.

In particular, for $n = 0$, frequency eqn (36) has only three complex roots and the root-loci plots are rather simple. For the couple that are axisymmetric about the imaginary axis, with the increase of ρ_0 or c_0, the absolute value of the real component of Ω decreases while the imaginary component increases; with the increase of e, the absolute value of the real component of Ω increases while the imaginary component decreases; with the increase of v, the absolute value of the real component of Ω increases while the imaginary component keeps basically invariant. For the pure imaginary, the effects of ρ_0, v and e are very weak, however, with the increase of c_0, the pure imaginary increases evidently. When $n \geqslant 1$, the number of root of frequency eqn (38) is greater than or equal to six and the root-loci plot is consequently more complicated. It is worth mentioning that, when $n = 1$, zero is a double root of frequency eqn (38). Numerical results show that frequency eqn (38) has at least a couple of roots with small absolute value of the real component as well as small imaginary part. This lower frequency with small damping coefficient is surely of practical importance.

5. SMALL DAMPING COEFFICIENT METHOD

In two previous papers (Lou and Su, 1978; Su, 1982), without obtaining the polynomial form frequency equations, Lou and Su thought that the frequency equations were very difficult to be exactly solved due to the inclusion of the second kind of spherical Hankel functions and, on account that designers in practical engineering were only interested in the frequencies with a small damping characteristic, a small damping coefficient method (SDC method) was developed to seek such frequencies, i.e.

$$\Omega = \Omega^0(1+\mathrm{i}\varepsilon), \tag{44}$$

where $\Omega^0 \geqslant 0$, $0 \leqslant \varepsilon \ll 1$. Substituting eqn (44) into the frequency equations, by virtue of Taylor expansion theorem, dropping all higher terms of ε, setting both the real and imaginary parts of the resulted equation equal to zero and eliminating ε, one can get a real coefficient equation of variable Ω^0. When Ω^0 is obtained from this equation, one can directly get ε and then Ω is also obtained. Details on this method can be found in Lou and Su (1978). Recently, authors (Ding and Chen, 1996b) also adopted the SDC method to solve their complicated and complex transcendental frequency equations.

How precise is the of SDC method? What restrictions should be imposed on it? Is it suitable for every case? These equations are still obscure. Therefore, it is very meaningful to check this method by comparing it with the present exact one. Results of two parameters by two methods are listed in Table 1.

It is shown from Table 1 that, for the first group of parameters, due to the evident damping effect of the fluid on the free vibration of the shell, the frequency equation itself has no root possessing small imaginary part (except for the case that $n = 1$, for which the trivial solution $\Omega = 0$ emerges) and the results obtained by SDC method deviate significantly from those obtained by the exact method. For the second group, however, the situation is opposite, and the results of SDC method agree well with the exact ones. In fact, by making a thorough investigation of the results, it can be found that for the first group of parameters, though the relative error of the imaginary part that represents the damping characteristic comes up as high as 125.8% ($n = 2$), the one of the real part that represents the speed of the intrinsic vibration of the submerged shell, which is most important in practical engineering, is just only of 14.1% ($n = 2$). This point shows that the results for problems with large damping factors obtained by SDC method are also valuable for reference in a certain degree. Especially for complicated, complex frequencies, SDC method can yet be regarded

Table 1. Comparison between SDC method and exact method (Ω)

	$\rho_0 = 0.2$, $c_0 = 0.2$, $e = 0.03$, $\nu = 0.3$ (1)		$\rho_0 = 0.05$, $c_0 = 0.5$, $e = 0.03$, $\nu = 0.3$ (2)	
n	SDC	EXACT	SDC	EXACT
0	(2.506, 0.662)	(2.591, 0.663)	(2.586, 0.401)	(2.616, 0.402)
1	(0, 0)	(0, 0)	(0, 0)	(0, 0)
2	(0.494, 0.359)	(0.433, 0.159)	(0.930, 0.068)	(0.935, 0.062)
3	(0.607, 0.231)	(0.599, 0.130)	(1.137, 0.029)	(1.143, 0.027)
4	(0.738, 0.135)	(0.747, 0.094)	(1.275, 0.007)	(1.279, 0.006)

as an effective means to seek the approximate periods of the natural vibration of the structure.

6. CONCLUSION

(1) The nonaxisymmetric free vibrations of a complete spherical shell submerged in a compressible fluid medium can be divided into two classes. In the absence of ambient fluid, the two classes of frequency equation are coincident with those obtained in Love (1944) with $e \to 0$.
(2) The second class of vibration of a submerged spherical shell is damping except for the case $n = 1$, for which one has $\Omega = 0$ and this corresponds to a rigid movement.
(3) The second class of frequency equation can be simplified to a complex algebraic equation of higher degree and further to a real one. In this sense, calculation is simplified.
(4) The introduction of polynomial $A_n(x)$ and the derivation of its recurrence formula make it convenient and direct to express the spherical Hankel functions. It can be seen that it is superior to the famous Rayleigh formula in the sense of application.
(5) When $n \geqslant 2$, it is found that frequency eqn (38) has at least one lower root possessing a weak damp characteristic in our calculations, which is of great importance in practical engineering.
(6) By comparing SDC method with the present exact one, it is shown that the error of SDC method is mainly shown in the imaginary part. However, from the point of view of practical consideration, the real part is still valuable for reference.

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